# Métodos Matemáticos de Bioingeniería

## Grado en Ingeniería Biomédica

## Chapter 2: Differentation in Several Variables

Lecture 5

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# Outline

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1. Functions of Several Variables; Graphing Surfaces
   - Functions and applications
   - Graphing functions: contour and level curves
   - Conic sections curves

## Motivation

- The volume and surface area of a sphere depend on its radius:

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad S = 4\pi r^2$$

- These equations define the volume and surface area as functions of the radius,

$$V(r) = \frac{4}{3}\pi r^3 \quad \text{and} \quad S(r) = 4\pi r^2$$

- Functions have an essential characteristic:

The so-called independent variable (the radius) determines a **unique** value of the dependent variable ($V$ or $S$)

## Motivation

- There are many quantities that are determined uniquely not by one variable **but by several**
    - The area of a rectangle
    - The volume of a cylinder or cone
    - The average rainfall in Madrid
    - The National debt
    - ...
- Realistic modelling of the world requires the understanding of:
    - The concept of a function of **more than one variable**
    - How to find meaningful ways to **visualize such functions**

## Features of Any Function

- Any function has three features
  1. A domain set $X$.
  2. A codomain set $Y$.
  3. A rule of assignment that associates to each element $x$ in the domain $X$ a unique element, $f(x)$, in the codomain $Y$.
- We will frequently use the notation

$$f : X \to Y$$

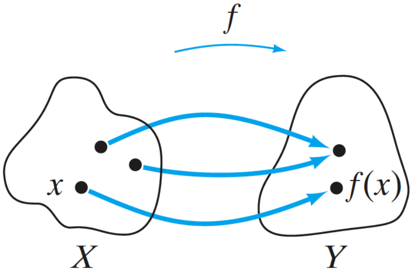
- Such notation indicates the sets involving a particular function

Although it does not make
the nature of the rule of assignment explicit

## Features of Any Function

$$f : X \to Y$$

- This notation also suggests the mapping nature of a function



## Example 1a

- Consider the act of assigning to each U.S. citizen his or her social security number.
- This pairing defines a function.

Each citizen is assigned
one social security number

- The **domain** is the set of U.S. citizens.
- The **codomain** is the set of all nine-digit strings of numbers.

## Example 1b

- A university assigns students to dormitory rooms.
- It is unlikely that it is creating a function from the set of available rooms to the set of students.
- Some rooms may have more than one student assigned to them.

A particular room does not necessarily determine a unique student occupant

### Definition 1.1: Range of a function

- The range or image of a function $f : X \to Y$ is the set of those elements of Y that are actual values of $f$.
- The range of $f$ consists of those $y$ in $Y$ such that $y = f(x)$ for some $x$ in $X$.
- Using set notation,

$$\text{Range f} = \text{Image f} = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$$

## Example 1a

- Recall the social security function of Example 1a.
- The range consists of those nine-digit numbers **actually** used as social security numbers.
- For example, is the number $000 - 00 - 0000$ in the range?

No one is actually
assigned this number

## Single-Variable Real functions

- For single-variable calculus, the functions of interest are those whose domains and codomains are subsets of $\mathbb{R}$.

Usually only the rule of assignment
is made explicit

- It is generally assumed that the domain is the largest possible subset of $\mathbb{R}$ for which the function makes sense.
- The codomain is generally taken to be all of $\mathbb{R}$.

## Example 3

- Suppose $g$ is a function such that $g(x) = \sqrt{x-1}$.
- If we take the codomain to be all of $\mathbb{R}$, the domain cannot be any larger than $[1, \infty)$.
- If the domain included any values less than one, the radicand would be negative and, hence, $g$ would not be real-valued.

## Multiple-Variable Real functions

- Multiple-variable real functions are the functions whose
  - Domains are subsets $X$ of $\mathbb{R}^n$, and
  - Codomains are subsets of $\mathbb{R}^m$, for some $n, m \in \mathbb{Z}^+$
- For simplicity of notation, we will take the codomains to be all of $\mathbb{R}^m$ (except when specified otherwise)
- Such a function is a mapping $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$

It associates to a vector (or point) $\mathbf{x}$ in $X$
a unique vector (point) $\mathbf{f}(\mathbf{x})$ in $\mathbb{R}^m$

## Example 4

- Let $T : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$T(x, y, z) = xy + xz + yz$$

- We can think of $T$ as a sort of "temperature function".
- Given a point $\mathbf{x} = (x, y, z)$ in $\mathbb{R}^3$,
  $T(x)$ calculates the temperature at that point.

## Example 5

- Let $L : \mathbb{R}^n \to \mathbb{R}$ be given by

$$L(\mathbf{x}) = \|\mathbf{x}\|$$

- This is a "length function".
- It computes the length of any vector $\mathbf{x}$ in $\mathbb{R}^n$.
- It is **one-one**? $L$ **is not**, since,

$$L(\mathbf{e}_i) = L(\mathbf{e}_j) = 1,$$

  with $\mathbf{e}_i$ and $\mathbf{e}_j$ any two of the standard basis vectors for $\mathbb{R}^n$.
- Is $L$ **onto**? $L$ also fails to be onto, since the length of a vector is always non-negative.

## Example 6

- Consider the function given by

$$\mathbf{N}(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$

where $\mathbf{x}$ is a vector in $\mathbb{R}^3$.

- Note that $\mathbf{N}$ is not defined if $\mathbf{x} = \mathbf{0}$, so the largest possible domain for $\mathbf{N}$ is,

$$\mathbb{R}^3 - \{\mathbf{0}\}$$

- The range of $\mathbf{N}$ consists of all unit vectors in $\mathbb{R}^3$.
- The function $\mathbf{N}$ is the "normalization function".
- It takes a nonzero vector in $\mathbb{R}^3$ and returns the unit vector that points in the same direction.

## Example 7

- Sometimes a function may be given numerically by a table.
- One such example is the notion of windchill.

The apparent temperature one feels when taking into account both the actual air temperature and the speed of the wind

| Air Temp (deg F) | Windspeed (mph) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 |
| 40 | 36 | 34 | 32 | 30 | 29 | 28 | 28 | 27 | 26 | 26 | 25 | 25 |
| 35 | 31 | 27 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | 19 | 18 | 17 |
| 30 | 25 | 21 | 19 | 17 | 16 | 15 | 14 | 13 | 12 | 12 | 11 | 10 |
| 25 | 19 | 15 | 13 | 11 | 9 | 8 | 7 | 6 | 5 | 4 | 4 | 3 |
| 20 | 13 | 9 | 6 | 4 | 3 | 1 | 0 | −1 | −2 | −3 | −3 | −4 |
| 15 | 7 | 3 | 0 | −2 | −4 | −5 | −7 | −8 | −9 | −10 | −11 | −11 |
| 10 | 1 | −4 | −7 | −9 | −11 | −12 | −14 | −15 | −16 | −17 | −18 | −19 |
| 5 | −5 | −10 | −13 | −15 | −17 | −19 | −21 | −22 | −23 | −24 | −25 | −26 |
| 0 | −11 | −16 | −19 | −22 | −24 | −26 | −27 | −29 | −30 | −31 | −32 | −33 |
| −5 | −16 | −22 | −26 | −29 | −31 | −33 | −34 | −36 | −37 | −38 | −39 | −40 |
| −10 | −22 | −28 | −32 | −35 | −37 | −39 | −41 | −43 | −44 | −45 | −46 | −48 |
| −15 | −28 | −35 | −39 | −42 | −44 | −46 | −48 | −50 | −51 | −52 | −54 | −55 |
| −20 | −34 | −41 | −45 | −48 | −51 | −53 | −55 | −57 | −58 | −60 | −61 | −62 |
| −25 | −40 | −47 | −51 | −55 | −58 | −60 | −62 | −64 | −65 | −67 | −68 | −69 |
| −30 | −46 | −53 | −58 | −61 | −64 | −67 | −69 | −71 | −72 | −74 | −75 | −76 |
| −35 | −52 | −59 | −64 | −68 | −71 | −73 | −76 | −78 | −79 | −81 | −82 | −84 |
| −40 | −57 | −66 | −71 | −74 | −78 | −80 | −82 | −84 | −86 | −88 | −89 | −91 |
| −45 | −63 | −72 | −77 | −81 | −84 | −87 | −89 | −91 | −93 | −95 | −97 | −98 |

## Example 7

| Air Temp (deg F) | Windspeed (mph) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **5** | **10** | **15** | **20** | **25** | **30** | **35** | **40** | **45** | **50** | **55** | **60** |
| 40 | 36 | 34 | 32 | 30 | 29 | 28 | 28 | 27 | 26 | 26 | 25 | 25 |
| 35 | 31 | 27 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | 19 | 18 | 17 |
| 30 | 25 | 21 | 19 | 17 | 16 | 15 | 14 | 13 | 12 | 12 | 11 | 10 |
| 25 | 19 | 15 | 13 | 11 | 9 | 8 | 7 | 6 | 5 | 4 | 4 | 3 |
| 20 | 13 | 9 | 6 | 4 | 3 | 1 | 0 | −1 | −2 | −3 | −3 | −4 |
| 15 | 7 | 3 | 0 | −2 | −4 | −5 | −7 | −8 | −9 | −10 | −11 | −11 |
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- If the air temperature is 20°F and the windspeed is 25 mph, the windchill temperature ("how cold it feels") is 3°F

## Example 7

| Air Temp (deg F) | Windspeed (mph) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 |
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- If $s$ denotes **windspeed** and $t$ **air temperature**, then the windchill is a function

$$W(s, t)$$

## Scalar-valued functions

- The functions described in Examples 4, 5, and 7 are scalar-valued functions

Functions whose codomains are
$\mathbb{R}$ or subsets of $\mathbb{R}$

- Scalar-valued functions are our main concern for this chapter.

- Nonetheless, let's look at a few examples of functions whose codomains are $\mathbb{R}^m$ where $m > 1$. They are usually called

Vector-valued functions

## Example 8

- Define $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$ by

$$\mathbf{f}(t) = (\cos t, \sin t, t)$$

- The range of $\mathbf{f}$ is the curve in $\mathbb{R}^3$ with parametric equations

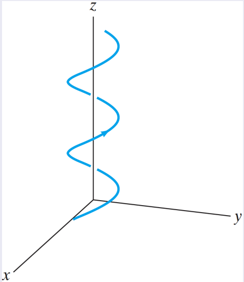
$$\begin{cases} x = \cos t \\ y = \sin t \qquad t \in \mathbb{R} \\ z = t \end{cases}$$

## Example 8

- Define $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$ by

$$\mathbf{f}(t) = (\cos t, \sin t, t)$$

- If we think of $t$ as a time parameter, then this function traces out the corkscrew curve.




- This curve is called a helix.

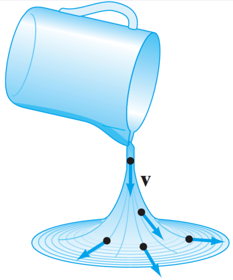
## Example 9

- We can think on the velocity of a fluid as a vector in $\mathbb{R}^3$
- This vector depends on (at least)
  - The point at which one measures the velocity , and
  - The time at which one makes the measurement
- Velocity may be considered to be a function

$$\mathbf{v} : X \subseteq \mathbb{R}^4 \to \mathbb{R}^3$$

- The domain $X$ is a subset of $\mathbb{R}^4$:
  - Three variables $x, y, z$ are required to describe a point in the fluid.
  - A fourth variable $t$ is needed to keep track of time.

## Example 9




- For instance, such a function **v** might be given by the expression

$$\mathbf{v}(x, y, z, t) = xyzt\mathbf{i} + (x^2 - y^2)\mathbf{j} + (3z + t)\mathbf{k}$$

## Vector-valued functions explicit form

- In general, if we have a function $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, then
  - $\mathbf{x} \in X$ can be written as,

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

  - and $\mathbf{f}$ can be written in terms of its component functions,

$$f_1, f_2, \ldots, f_m$$

- The component functions are scalar-valued functions of $\mathbf{x} \in X$.
- So, **Vector functions** can be written as the Cartesian product of scalar-valued functions. In general, is enough to study the properties of the scalar-valued function, and then apply those properties in each component.

## Vector-valued functions explicit form

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(x_1, x_2, \ldots, x_n)$$

(emphasizing the variables)

$$= (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$$

(emphasizing the component functions)

$$= (f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n))$$

(writing out all components)

### Example 5

- Let $L : \mathbb{R}^n \to \mathbb{R}$ be given by

$$L(\mathbf{x}) = \|\mathbf{x}\|$$

- The function $L$, when expanded, becomes

$$L(\mathbf{x}) = L(x_1, x_2, \ldots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

## Example 6

- Consider the function given by

$$\mathbf{N}(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$

where $\mathbf{x}$ is a vector in $\mathbb{R}^3$ .

- The function $\mathbf{N}$ becomes

$$\begin{aligned}\mathbf{N}(\mathbf{x}) &= \frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{(x_1, x_2, x_3)}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\ &= \left( \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \right)\end{aligned}$$

## Example 6

- Hence, the three component functions of **N** are

$$\begin{aligned} N_1(x_1, x_2, x_3) &= \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\ N_2(x_1, x_2, x_3) &= \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\ N_3(x_1, x_2, x_3) &= \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \end{aligned}$$

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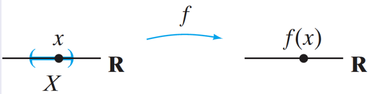
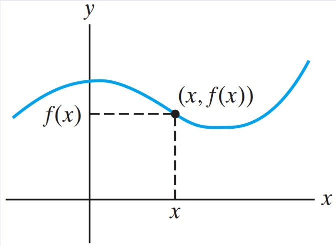
## Graphing Scalar-Valued Functions

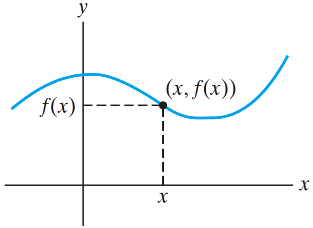
- A function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ takes a real number and returns another real number




- The graph of $f$ is something that "lives" in $\mathbb{R}^2$



## Graphing Scalar-Valued Functions




- The graph of $f$ consists of points $(x, y)$ such that $y = f(x)$.

$$\text{Graph } f = \{(x, f(x)) \mid x \in X\}$$

- In general, the graph of a scalar-valued function of a single variable is a curve

A one-dimensional object
sitting inside a two dimensional space

## Graphing a Function of Two Variables

- Suppose we have a function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$
- We make essentially the same definition for the graph

$$\text{Graph } f = \{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x} \in X\}$$

$$\mathbf{x} = (x, y) \text{ is a point of } \mathbb{R}^2$$

- Thus, $\{(\mathbf{x}, f(\mathbf{x}))\}$ may also be written as

$$\{(x, y, f(x, y))\} \quad \text{or} \quad \{(x, y, z) \mid (x, y) \in X, \; z = f(x, y)\}$$

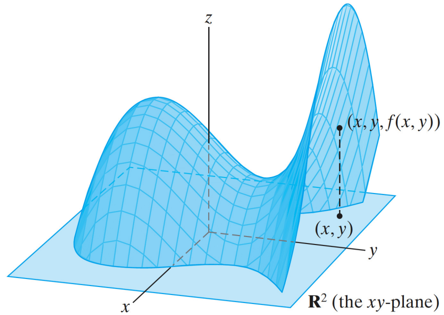
- Hence, the graph of a scalar-valued function of two variables is something that sits in $\mathbb{R}^3$

The graph will be a surface

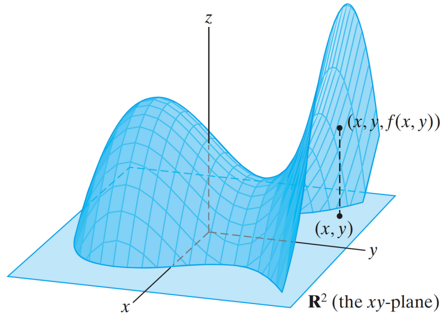
## Example 10

- Consider the graph of the function

$$f:\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=\frac{1}{12}y^3-y-\frac{1}{4}x^2+\frac{7}{2}$$



## Example 10




- Each point $\mathbf{x} = (x, y)$ in $\mathbb{R}^2$, is graphed as a point in $\mathbb{R}^3$ with coordinates

$$\left(x, y, \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2}\right)$$

## Contour Curves and Level Curves

- Graphing functions of two variables is a much more difficult task than graphing functions of one variable

One method is
to let a computer do the work

- Nonetheless, to get a feeling, a sketch of a rough graph is still a valuable skill

The trick is to find a way
to cut down on the dimensions involved

- One way is to draw certain special curves that lie on the surface

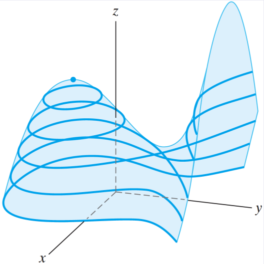
$$z = f(x, y)$$

## Contour Curves and Level Curves

- This special curves that lie on the surface $z = f(x, y)$ are called contour curves.
- Contour curves are obtained by intersecting the surface with horizontal planes $z = c$, for various values of the constant $c$.

## Example 10

- Some contour curves drawn on the surface of Example 10



## Contour Curves and Level Curves

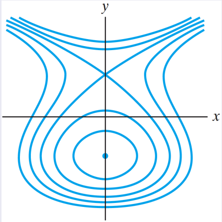
- Let us compress all the contour curves onto the $xy$-plane

Look down
along the positive $z$-axis

- Then we create a "topographic map" of the surface called level curves of the original function $f$

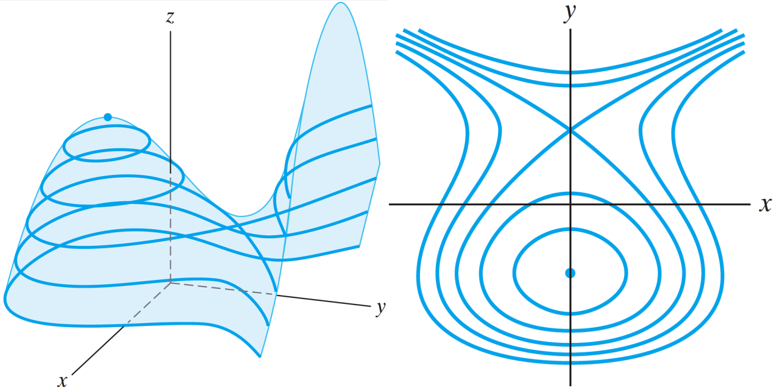
## Example 10

- Some level curves drawn for the Example 10



## Example 10

- Some contour and level curves drawn for the Example 10



## Contour Curves and Level Curves

We can reverse the process in order to sketch systematically the graph of a function $f$ of two variables

1. We first construct a topographic map in $\mathbb{R}^2$ by finding the **level curves** of $f$.
2. Then situate these curves in $\mathbb{R}^3$ as **contour curves** at the appropriate heights.
3. Finally, complete the graph of the function.

## Definition 1.4: Level Curve

- Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a scalar-valued function of two variables.
- The level curve at height $c$ of $f$ is the curve in $\mathbb{R}^2$ defined by the equation
$$f(x, y) = c$$
where $c$ is a constant.
- In mathematical notation,
$$L_c = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$$

## Definition 1.4: Contour Curve

- Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a scalar-valued function of two variables
- The contour curve at height $c$ of $f$ is the level curve drawn in $\mathbb{R}^3$.
- In mathematical notation,

$$C_c = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y) = c\}$$

## Example 11

Use level and contour curves to construct the graph of the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = 4 - x^2 - y^2$$

- By Definition 1.4, the level curve at height $c$ is

$$\{(x, y) \in \mathbb{R}^2 \mid 4 - x^2 - y^2 = c\} = \{(x, y) \mid x^2 + y^2 = 4 - c\}$$

- The level curves for $c < 4$ are circles centered at the origin of radius $\sqrt{4 - c}$
- The level "curve" at height $c = 4$ is not a curve but just a single point (the origin)
- There are no level curves at heights larger than 4, since the equation $x^2 + y^2 = 4 - c$ has no real solutions in $x$ and $y$

## Example 11

Use level and contour curves to construct the graph of the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = 4 - x^2 - y^2$$

| $c$ | level curve $x^2 + y^2 = 4 - c$ |
|---|---|
| $-5$ | $x^2 + y^2 = 9$ |
| $-1$ | $x^2 + y^2 = 5$ |
| $0$ | $x^2 + y^2 = 4$ |
| $1$ | $x^2 + y^2 = 3$ |
| $3$ | $x^2 + y^2 = 1$ |
| $4$ | $x^2 + y^2 = 0 \iff x = y = 0$ |
| $c$, where $c > 4$ | empty |

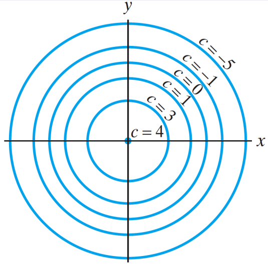
## Example 11

Use level and contour curves to construct the graph of the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = 4 - x^2 - y^2$$

- “Topographic map” or family of level curves of the surface

$$z = 4 - x^2 - y^2$$

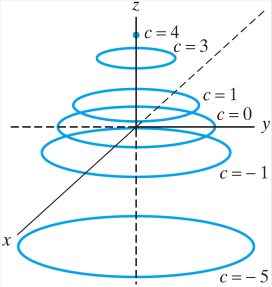


## Example 11

Use level and contour curves to construct the graph of the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = 4 - x^2 - y^2$$

- Some contour curves, which sit in $\mathbb{R}^3$:

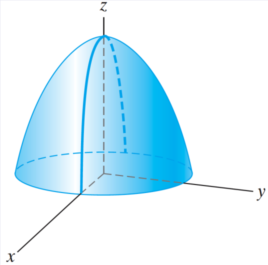


## Example 11

Use level and contour curves to construct the graph of the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = 4 - x^2 - y^2$$

- Complete graph of surface




- It looks like an inverted antenna and is called a paraboloid

## Example 11 - Sections of the surface

If you want to get even a better feeling about the shape of the surface, is also useful to "chop" the surface with (vertical) planes of the form $x = c$ or $y = c$:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = 4 - x^2 - y^2$$

- Section of the surface $z = 4 - x^2 - y^2$ by the plane $x = 0$

$$\{(x, y, z) \in \mathbb{R}^3 \mid z = 4 - x^2 - y^2, x = 0\} = \{(0, y, z) \mid z = 4 - y^2\}$$

- Section of the surface $z = 4 - x^2 - y^2$ by the plane $y = 0$

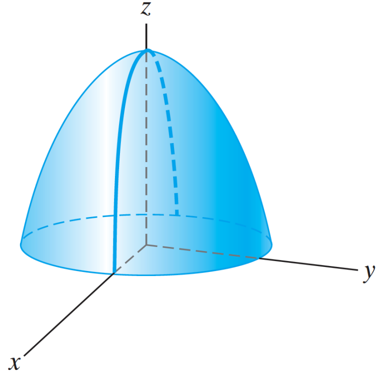
$$\{(x, y, z) \in \mathbb{R}^3 \mid z = 4 - x^2 - y^2, y = 0\} = \{(x, 0, z) \mid z = 4 - x^2\}$$

All these sections are parabolas

## Example 11

Use level and contour curves to construct the graph of the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = 4 - x^2 - y^2$$



## Example 12

Graph the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = y^2 - x^2$$

| $c$ | level curve $y^2 - x^2 = c$ |
|---|---|
| $-4$ | $x^2 - y^2 = 4$ |
| $-1$ | $x^2 - y^2 = 1$ |
| $0$ | $y^2 - x^2 = 0 \iff (y - x)(y + x) = 0 \iff y = \pm x$ |
| $1$ | $y^2 - x^2 = 1$ |
| $4$ | $y^2 - x^2 = 4$ |

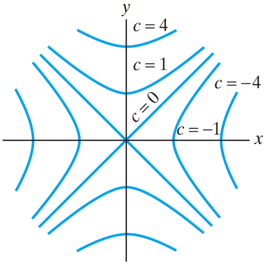
## Example 12

Graph the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = y^2 - x^2$$

The level curves are almost all hyperbolas

- The exception is the level curve at height 0, which is a pair of intersecting lines



## Example 12

Graph the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = y^2 - x^2$$

- Section of the surface $z = y^2 - x^2$ by the plane $x = c$

$$\{(x,y,z) \in \mathbb{R}^3 \mid z = y^2 - x^2, x = c\} = \{(c,y,z) \mid z = y^2 - c^2\}$$

Parabolas in the planes $x = c$

- Section of the surface $z = y^2 - x^2$ by the plane $y = c$

$$\{(x,y,z) \in \mathbb{R}^3 \mid z = y^2 - x^2, y = c\} = \{(c,y,z) \mid z = c^2 - x^2\}$$

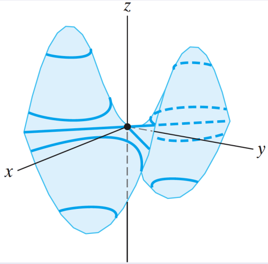
Parabolas in the planes $y = c$

## Example 12

Graph the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = y^2 - x^2$$

- The level curves and sections generate the contour curves and surface depicted in figure




- This surface is called a hyperbolic paraboloid.

## Example 13

Graph the function:

$$h : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}, \quad h(x,y) = \ln(x^2 + y^2)$$

- The level curve of $h$ at height $c$ is

$$\{(x,y) \in \mathbb{R}^2 \mid \ln(x^2 + y^2) = c\} = \{(x,y) \mid x^2 + y^2 = e^c\}$$

- Since $e^c > 0$ for all $c \in \mathbb{R}$, the level curve
  - Exists for any $c$, and
  - Is a circle of radius $\sqrt{e^c} = e^{c/2}$

## Example 13

Graph the function:

$$h : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}, \quad h(x,y) = \ln(x^2 + y^2)$$

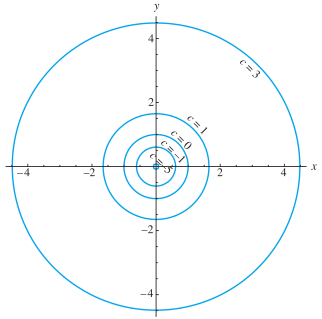
| $c$ | level curve $x^2 + y^2 = e^c$ |
|---|---|
| $-5$ | $x^2 + y^2 = e^{-5}$ |
| $-1$ | $x^2 + y^2 = e^{-1}$ |
| $0$ | $y^2 + x^2 = 1$ |
| $1$ | $y^2 + x^2 = e$ |
| $3$ | $y^2 + x^2 = e^3$ |
| $4$ | $y^2 + x^2 = e^4$ |

## Example 13

Graph the function:

$$h : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}, \quad h(x,y) = \ln(x^2 + y^2)$$

- The collection of level curves is shown in figure



## Example 13

Graph the function:

$$h : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}, \quad h(x,y) = \ln(x^2 + y^2)$$

- Section of the graph by the plane $x = 0$

$$\begin{aligned}\{(x,y,z) \in \mathbb{R}^3 \mid z = \ln(x^2 + y^2), x = 0\} \\ = \{(0,y,z) \mid z = \ln(y^2) = 2\ln|y|\}\end{aligned}$$

- Section of the graph by the plane $y = 0$

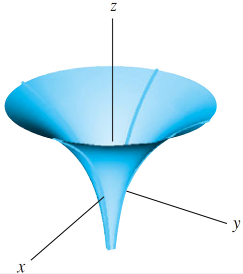
$$\begin{aligned}\{(x,y,z) \in \mathbb{R}^3 \mid z = \ln(x^2 + y^2), y = 0\} \\ = \{(x,0,z) \mid z = \ln(x^2) = 2\ln|x|\}\end{aligned}$$

## Example 13

Graph the function:

$$h : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}, \quad h(x,y) = \ln(x^2 + y^2)$$

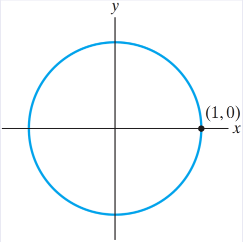
- The complete graph is shown in figure



## The Unit Circle Example

Important remark:
Not all curves in $\mathbb{R}^2$ can be described as
the graph of a single function of one variable

- The most familiar example is the unit circle




- Its graph cannot be determined by a single equation of the form

$$y = f(x)$$

## The Unit Circle Example

- The graph of the circle may be described analytically by the equation

$$x^2 + y^2 = 1$$

- In general, a curve in $\mathbb{R}^2$ is determined by an arbitrary equation in $x$ and $y$

Not necessarily one that
isolates $y$ alone on one side

- This means that a general curve is given by an equation of the form

$$F(x, y) = c$$

As level set of
a function of two variables

## The Unit Circle Example

- The analogous situation occurs with surfaces in $\mathbb{R}^3$.
- Frequently a surface is determined by an equation of the form

$$F(x, y, z) = c$$

As a level set of
a function of three variables

- It is not necessarily one of the form

$$z = f(x, y)$$

If it is not possible to state a variable as function of the other ones then is called a **implicit formula**. All functions can be expressed as implicit formula doing, $z - f(x, y) = c, \ c = 0$.

## Example 15

A sphere is a surface in $\mathbb{R}^3$
whose points are all equidistant from a fixed point

- If this fixed point is the origin, then the equation for the sphere is

$$\|\mathbf{x} - \mathbf{0}\| = \|\mathbf{x}\| = a$$

where $a$ is a positive constant and $\mathbf{x} = (x, y, z)$ is a point on the sphere.

- If we square both sides of equation and expand the dot product

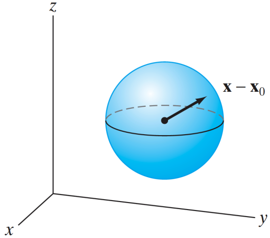
$$x^2 + y^2 + z^2 = a^2$$

## Example 15

A sphere is a surface in $\mathbb{R}^3$
whose points are all equidistant from a fixed point

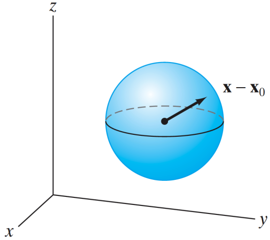
- If the center of the sphere is at the point $\mathbf{x_0} = (x, y, z)$ then equation should be modified to

$$\|\mathbf{x} - \mathbf{x_0}\| = a$$



## Example 15

A sphere is a surface in $\mathbb{R}^3$
whose points are all equidistant from a fixed point




- When equation $\|\mathbf{x} - \mathbf{x_0}\| = a$ is expanded, the following general equation for a sphere is obtained

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

## Example 15

A sphere is a surface in $\mathbb{R}^3$
whose points are all equidistant from a fixed point

$$x^2 + y^2 + z^2 = a^2$$

- In the equation for a sphere, there is no way to solve for $z$ uniquely in terms of $x$ and $y$
- If we try to isolate $z$, then

$$z^2 = a^2 - x^2 - y^2$$

- So we are forced to make a choice of positive or negative square roots in order to solve for $z$

$$z = \sqrt{a^2 - x^2 - y^2} \quad \text{or} \quad z = -\sqrt{a^2 - x^2 - y^2}$$

## Example 15

A sphere is a surface in $\mathbb{R}^3$
whose points are all equidistant from a fixed point

$$z = \sqrt{a^2 - x^2 - y^2} \quad \text{or} \quad z = -\sqrt{a^2 - x^2 - y^2}$$

- The positive square root corresponds to the upper hemisphere.
- The negative square root corresponds to the lower hemisphere.

In any case, the entire sphere cannot be the graph of
a single function of two variables

# Outline

# Quadratic Surfaces

## Conic sections

- Conic sections are curves obtained from the intersection of a cone with various planes.
- They are among the simplest, yet also the most interesting, of plane curves:
  - The circle.
  - The ellipse.
  - The parabola.
  - The hyperbola.
- They have an elegant algebraic connection:

Every conic section is described analytically by
a polynomial equation of degree two in two variables

## Conic sections

Every conic section is described analytically by
a polynomial equation of degree two in two variables

- That is, every conic can be described by an equation that looks like

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

for suitable constants $A, \ldots, F$.

## General Quadric Surfaces

- In $\mathbb{R}^3$, the analytic analogue of the conic section is called a quadratic surface.
- Quadratic surfaces are defined by equations that are polynomials of degree two in three variables

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + Gx + Hy + Iz + J = 0$$

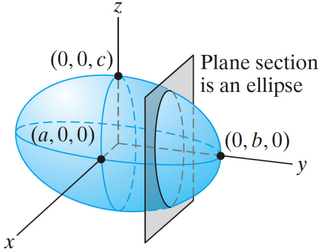
- To pass from this equation to the appropriate graph is, in general, a cumbersome process

  We need the aid of either a computer or more linear algebra than we currently have at our disposal

- We sketch some examples whose corresponding analytic equations are relatively simple.

## Ellipsoid

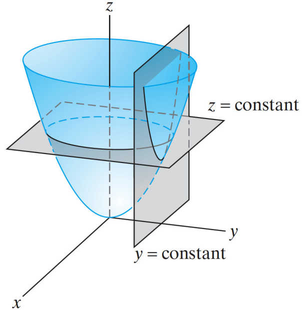
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$




- It is the three-dimensional analogue of an ellipse in the plane.
- If $a = b = c$, then the ellipsoid is a sphere of radius $a$.
- The sections of the ellipsoid by planes perpendicular to the coordinate axes are all ellipses.

## Elliptic paraboloid

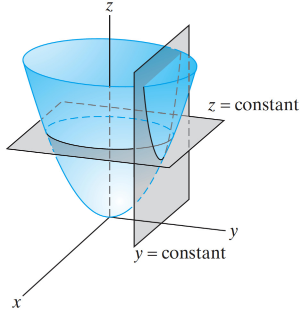
$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$




- The paraboloid has:
  - Elliptical (or single-point or empty) sections by the planes "$z$ = constant", and
  - Parabolic sections by "$x$ = constant" or "$y$ = constant" planes

## Elliptic paraboloid

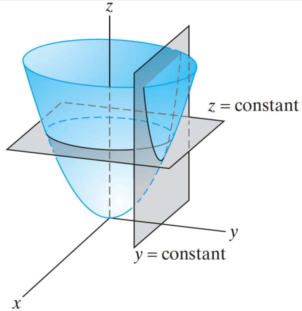
$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$




- The constants $a$ and $b$ affect the aspect ratio of the elliptical cross sections

## Elliptic paraboloid

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

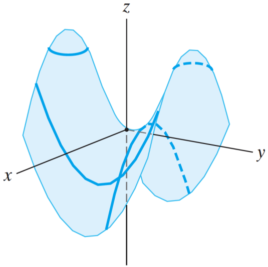



- The constant $c$ affects the steepness of the dish

Larger values of $c$
produce steeper paraboloids

## Hyperbolic paraboloid

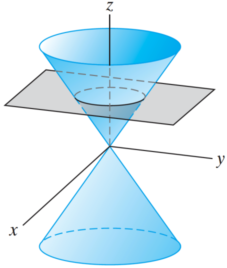
$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$




- It is shaped like a **saddle whose**
  - "$x =$ constant" or "$y =$ constant" sections are parabolas , and
  - "$z =$ constant" sections are hyperbolas

## Elliptic cone

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$




- The sections by "$z =$ constant" planes are ellipses
- The sections by $x = 0$ or $y = 0$ are each a pair of intersecting lines.